

Universal R operator with Jordanian deformation of conformal symmetry

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Abstract

The Jordanian deformation of $sl(2)$ bi-algebra structure is studied in view of physical applications to breaking of conformal symmetry in the high energy asymptotics of scattering. Representations are formulated in terms of polynomials, generators in terms of differential operators. The deformed R operator with generic representations is analyzed in spectral and integral forms.

1 Introduction

In the last decade high-energy scattering in gauge field theories has become a new area of application of integrable quantum systems [1, 2]. In several cases the effective interaction appearing in the Regge and the Bjorken asymptotics of scattering is determined by Hamiltonians of integrable chains, where the representations on the sites are infinite-dimensional. In a number of papers the methods of integrable systems have been reformulated and developed for the special needs of these new applications, e.g. [3, 4]. They have been applied to the renormalization of higher twist operators [5, 6]. Integrable structures have been encountered also in recent studies of composite operators in $\mathcal{N} = 4$ super Yang-Mills theory in the large N_C limit motivated by questions related to the AdS/CFT hypothesis [7, 8] .

In the context of integrable chains with non-compact representations on the sites it is convenient to consider wave functions describing the states on the sites. In the case of the

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Bjorken asymptotics the one-site wave function depends on one variable, the position on the light ray. Conformal transformation of the light ray is the relevant symmetry here, the underlying *Lie* algebra is $sl(2)$. Infinitesimal symmetry transformation of the wave functions act as differential operators.

Owing to these applications solutions of the quantum Yang-Baxter equation (QYBE) have been studied with generic representations of the symmetry algebras $sl(2)$, $sl(2|1)$ and the q -deformation of $sl(2)$ [9, 10, 11]. A scheme has been developed relying on known methods [12, 13, 14, 15] and resulting in formulations suitable for the mentioned applications.

In the present paper we treat along these lines the non-standard or Jordanian (ξ) deformation of $sl(2)$. Whereas there is an extensive literature on the standard q deformation of Lie bi-algebra structure and on its application, the case of ξ - deformation did not attract comparable attention so far, although the related literature is numerous and cannot be review here. Some early papers concerning this subject are [16, 17, 18, 19, 20, 21, 22, 23].

The concepts of Yangians and of their deformation by twist have been developed by Drinfeld [24, 25]. The basis commutation relation of ξ deformed $sl(2)$ algebra have been written first by Ohn [21]. The explicite form of the twist transformation appeared first in [23, 22]. The twist construction of the deformed Yangian has been described e.g. in [26, 27]. The structure of ξ - deformed representations has been studied e.g. in [28, 29]. Applications to spin chains have been discussed in [26] and to the modification of space-time structure e.g. in [30, 31, 32].

From the viewpoint of conformal symmetry representations the ξ deformation of $sl(2)$ leaves the generator of translation undisturbed but deforms the one of dilatation. This means momentum conservation holds, whereas scale symmetry is broken. By the deformation parameter a (length or momentum) scale is introduced. It remains to be investigated whether the pattern of scale symmetry breaking by ξ deformation, which will be worked out to some detail here, could be applied to describe the scale symmetry breaking effects in gauge field theories.

In the next section we summarize some relevant features of Jordanian deformation of $sl(2)$ algebraic structure. In section 3 the QYBE with R operators of the fundamental and generic representations are considered. In section 4 the realization of representations by polynomials and of generators by differential operators are formulated. Tensor product representations are formulated in terms of polynomials in section 5. The universal R operator in spectral and integral forms is considered in section 6.

2 Algebraic deformations

2.1 Deformations induced by classical Yang-Baxter solutions

We start from Yang's solution of QYBE with $sl(2)$ symmetry

$$R^{(0)}(u) = u I + \eta P = u \left(I + \eta r^{(0)}(u) \right) \quad (2.1)$$

with the corresponding solution of the classical Yang-Baxter equation (CYBE),

$$r^{(0)}(u) = \frac{1}{2u}(I + c_2), \quad c_2 = \frac{1}{2}(S_0^+ \otimes S_0^- + S_0^- \otimes S_0^+ + 2S_0^0 \otimes S_0^0), \quad (2.2)$$

and the Yangian algebraic structure [25, 24] associated to them. S_0^a are the $sl(2)$ generators obeying the standard commutation relations,

$$[S_0^0, S_0^\pm] = \pm S_0^\pm, \quad [S_0^+, S_0^-] = 2S_0^0. \quad (2.3)$$

Deformations of this structure are induced by spectral parameter independent solutions of CYBE. The standard q -deformation is induced by

$$r_q = S_0^+ \otimes S_0^- - S_0^- \otimes S_0^+ \quad (2.4)$$

This deformation can be characterized by the deformed generators obeying

$$[S_q^0, S_q^\pm] = \pm S_q^\pm, \quad [S_q^+, S_q^-] = [2S_q^0]_q, \quad (2.5)$$

with the notation $[a]_q = (q - q^{-1})^{-1}(q^a - q^{-a})$, and the deformed co-products, defined on them as

$$\Delta_q(S_q^\pm) = S_{q,1}^\pm q^{S_{q,2}^0} + S_{q,2}^\pm q^{-S_{q,1}^0}, \quad \Delta_q(S_q^0) = S_{q,1}^0 + S_{q,2}^0, \quad \bar{\Delta}_q = \Delta_{q^{-1}}. \quad (2.6)$$

The remarkable feature is here that the co-product for S^0 and also the commutators with it are not changed. From the point of view of representations of conformal symmetry in 1 dimension one can say that the q -deformation preserves dilatation symmetry but the symmetry of translation and proper-conformal transformations is broken.

The classification of solution of CYBE associated with a simple *Lie* algebra results in a classification of possible deformations [33]. In the case of $sl(2)$ there is a further interesting but not so well studied deformation induced by

$$r_\xi = S_0^0 \otimes S_0^- - S_0^- \otimes S_0^0, \quad (2.7)$$

built from the Borel subalgebra B_- generated by S_0^-, S_0^0 . This deformation can be characterized by the deformed generators obeying

$$\begin{aligned} [S_\xi^+, S_\xi^-] &= 2S_\xi^0, & [S_\xi^0, S_\xi^-] &= -\frac{1}{\xi} \text{sh}(\xi S_\xi^-), \\ [S_\xi^0, S_\xi^+] &= \frac{1}{2} \{ \text{ch}(\xi S_\xi^-), S_\xi^+ \}, \end{aligned} \quad (2.8)$$

and by the co-products defined on them as

$$\begin{aligned} \Delta_\xi(S_\xi^a) &= S_{\xi,12}^a = S_{\xi,1}^a e^{\xi S_{\xi,2}^-} + e^{-\xi S_{\xi,1}^-} S_{\xi,2}^a, \quad \text{for } a = 0, +, \\ \Delta_\xi(S_\xi^-) &= S_{\xi,12}^- = S_{\xi,1}^- + S_{\xi,2}^-, \quad \bar{\Delta}_\xi = \Delta_{(-\xi)}. \end{aligned} \quad (2.9)$$

Now the co-products of S^- are not changed. In the conformal symmetry representations S_0^- can play the role of the translation generator and one can say that the ξ deformation preserves translation symmetry whereas the symmetries of dilatation and proper-conformal transformations are now broken. In this context the deformation parameter ξ has the natural dimension of length, whereas the deformation parameter q is dimensionless.

The ξ -deformation of the Casimir operator, i.e. the element of the enveloping algebra commuting with all S_ξ^a and approaching the $sl(2)$ Casimir operator at $\xi \rightarrow 0$, has the form

$$C_\xi = \frac{2}{\xi} \text{sh} \frac{\xi S_\xi^-}{2} S_\xi^+ \text{ch} \frac{\xi S_\xi^-}{2} + S_\xi^0 (S_\xi^0 + 1) = \frac{2}{\xi} \text{ch} \frac{\xi S_\xi^-}{2} S_\xi^+ \text{sh} \frac{\xi S_\xi^-}{2} + S_\xi^0 (S_\xi^0 - 1) \quad (2.10)$$

The commutativity with S_ξ^a and the different forms are checked by using the following consequences of (2.8),

$$S_\xi^0 e^{\alpha \xi S_\xi^-} = e^{\alpha \xi S_\xi^-} (S_\xi^0 - \alpha \text{sh} \xi S_\xi^-), \quad (2.11)$$

and

$$S_\xi^+ e^{\alpha \xi S_\xi^-} = e^{\alpha \xi S_\xi^-} (S_\xi^+ + 2\alpha \xi S_\xi^0 - 2\alpha^2 \xi \text{sh} \xi S_\xi^-).$$

2.2 Drinfeld twist

The ξ -deformation can be described relying on a general result by Drinfeld [24].

Notice that $r_\xi \in B_- \otimes B_-$. There exists an element F_ξ in the enveloping algebra $U(B_-) \otimes U(B_-)$ such that

$$F_\xi^{12}((\Delta_0 \otimes id)F_\xi = F_\xi^{23}(id \otimes \Delta_0)F_\xi \quad (2.12)$$

(Δ_0 is the original co-commutative coproduct) with

$$F_\xi = I \otimes I + \xi r_\xi + \mathcal{O}(\xi^2), \quad (2.13)$$

and such that the deformed co-product can be written as

$$\Delta_\xi = F_\xi \Delta_0 F_\xi^{-1}. \quad (2.14)$$

If $R^{(0)}$ is a solution of QYBE related to the co-product Δ_0 then

$$R_{12}^{(\xi)} = F_{\xi,21} R_{12}^{(0)} F_{\xi,12}^{-1} \quad (2.15)$$

is a solution of QYBE related to Δ_ξ , namely

$$\Delta_\xi R^{(\xi)} = R^{(\xi)} \bar{\Delta}_\xi. \quad (2.16)$$

The twist element of the ξ -deformation of the $sl(2)$ Yangian can be written in terms of

$$\sigma_\xi = -\ln(I - 2\xi S_0^-)$$

as

$$F_\xi = \exp(S_0^0 \otimes \sigma_\xi), \quad F_{\xi,12} = (1 - 2\xi S_{0,2}^-)^{-S_{0,1}^0}. \quad (2.17)$$

Let us verify (2.12). Rewrite the relation in the sense of operators acting on $V_1 \otimes V_2 \otimes V_3$. Because

$$\begin{aligned} (\Delta \otimes id)F &= (1 - 2\xi S_{0,3}^-)^{-S_{0,1}^0 - S_{0,2}^0}, \\ (id \otimes \Delta)F &= (1 - 2\xi S_{0,2}^- - 2\xi S_{0,3}^-)^{-S_{0,1}^0} \end{aligned}$$

(2.12) takes the form

$$\begin{aligned} (1 - 2\xi S_{0,3}^-)^{S_{0,2}^0} (1 - 2\xi S_{0,2}^-)^{-S_{0,1}^0} (1 - 2\xi S_{0,3}^-)^{-S_{0,2}^0} (1 - 2\xi S_{0,3}^-)^{-S_{0,1}^0} = \\ (1 - 2\xi S_{0,2}^- - 2\xi S_{0,3}^-)^{-S_{0,1}^0} \end{aligned} \quad (2.18)$$

We notice that

$$A^{S_{0,2}^0} S_{0,2}^- A^{-S_{0,2}^0} = A^{-1} S_{0,2}^-,$$

provided A commutes with $S_{0,2}^-$, therefore

$$(1 - 2\xi S_{0,3}^-)^{S_{0,2}^0} (1 - 2\xi S_{0,2}^-)^{-S_{0,1}^0} (1 - 2\xi S_{0,3}^-)^{-S_{0,2}^0} = (1 - \frac{2\xi S_{0,2}^-}{1 - 2\xi S_{0,3}^-})^{-S_{0,1}^0} =$$

$$(1 - 2\xi S_{0,2}^- - 2\xi S_{0,3}^-)^{-S_{0,1}^0} (1 - 2\xi S_{0,3}^-)^{S_{0,1}^0},$$

which implies (2.18) and therefore (2.12).

3 From the fundamental to generic representations

3.1 Fundamental representation R matrix

We label the representations by the conformal weight ℓ , the eigenvalue of S^0 of the lowest weight state in the representation space. The finite-dimensional representations of spin and angular momentum correspond to negative half-integer and negative integer values of ℓ .

Representing the generators by *Pauli* matrices,

$$S_{\xi, -\frac{1}{2}}^+ = \hat{\sigma}^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_{\xi, -\frac{1}{2}}^- = \hat{\sigma}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_{\xi, -\frac{1}{2}}^0 = \frac{1}{2}\hat{\sigma}^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.1)$$

the effect of the deformation disappears in the commutation relations (2.8) and is only present in the co-product (2.9). We shall obtain the solution of QYBE in the fundamental representation $V_1^{-\frac{1}{2}} \otimes V_2^{-\frac{1}{2}}$. First we start from the direct implication of (2.16),

$$R_{12}^{(\xi)} S_{-\xi, 12}^a = S_{+\xi, 12}^a R_{12}^{(\xi)}. \quad (3.2)$$

The condition with $a = -$ implies that the 4×4 matrix representing R has the form

$$R^{(\xi)} = u \begin{pmatrix} u + \eta & & & \\ r_{21} & u & \eta & \\ r_{31} & \eta & u & \\ r_{41} & r_{42} & r_{43} & u + \eta \end{pmatrix}.$$

The up to now unknown matrix elements vanish at $\xi = 0$ and in this way the undeformed solution (2.1) is recovered. One of the remaining conditions, $a = 0$ or $a = +$, restricts the ansatz to the solution,

$$R^{(\xi), (-\frac{1}{2}, -\frac{1}{2})} = u \begin{pmatrix} 1 & & & \\ -\xi & 1 & & \\ \xi & & 1 & \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix} + \eta \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad (3.3)$$

and one checks easily that QYBE,

$$R_{b_1 b_2}^{a_1 a_2}(u - v) R_{c_1 b_3}^{b_1 a_3}(u - w) R_{c_2 c_3}^{b_2 b_3}(v - w) = R_{b_2 b_3}^{a_2 a_3}(v - w) R_{b_1 c_3}^{a_1 b_3}(u - w) R_{c_1 c_2}^{b_1 b_2}(u - v), \quad (3.4)$$

is fulfilled.

In the above arguments we did not use the full algebraic information available. Using the twist relation (2.15) allows to obtain this result easier. In the representation $V_1^{-\frac{1}{2}} \otimes V_2^{-\frac{1}{2}}$ the twist element (2.17) is calculated as

$$F_{\xi, 12} = (I - 2\xi S_{0,2}^-)^{-S_{0,1}^0} = I_{4 \times 4} + \xi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\xi & 1 \end{pmatrix}. \quad (3.5)$$

Applying the twist relation (2.15),

$$R_{12}^{(\xi), (-\frac{1}{2}, -\frac{1}{2})} = F_{\xi, 21} (u I_{4 \times 4} + \eta P_{4 \times 4}) F_{\xi, 12}^{-1} = u F_{\xi, 21} F_{\xi, 12}^{-1} + \eta P, \quad (3.6)$$

we recover the result (3.3).

3.2 Lax operator

We consider QYBE with the representations specified as the fundamental ones, $\ell_1 = \ell_2 = -\frac{1}{2}$, for the tensor factors 1 and 2 but generic, $\ell_3 = \ell$, for the 3rd factor and look for the expression for $R_{13}^{(\xi)}$ or $R_{23}^{(\xi)}$ as a 2×2 matrix with the elements being operators in V_3^ℓ ,

$$R^{(\xi), (-\frac{1}{2}, \ell)}(u) = L(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (3.7)$$

The $a = -$ component of the symmetry conditions (3.2) in the case $\ell_1 = -\frac{1}{2}$, $\ell_2 = \ell$ can be written explicitly as

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} S_\xi^- & 0 \\ 1 & S_\xi^- \end{pmatrix} = \begin{pmatrix} S_\xi^- & 0 \\ 1 & S_\xi^- \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad (3.8)$$

and implies

$$[b, S_\xi^-] = 0, \quad [a, S_\xi^-] = -b, \quad [d, S_\xi^-] = b, \quad [c, S_\xi^-] = a - d. \quad (3.9)$$

We write explicitly also the $a = 0$ component of (3.2)

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} S_\xi^0 + \frac{1}{2}e^{\xi S_\xi^-} & 0 \\ -\xi S_\xi^0 & S_\xi^0 - \frac{1}{2}e^{\xi S_\xi^-} \end{pmatrix} = \begin{pmatrix} S_\xi^0 + \frac{1}{2}e^{-\xi S_\xi^-} & 0 \\ \xi S_\xi^0 & S_\xi^0 - \frac{1}{2}e^{-\xi S_\xi^-} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \quad (3.10)$$

In particular this implies

$$b(S_\xi^0 - \frac{1}{2}e^{\xi S_\xi^-}) = (S_\xi^0 + \frac{1}{2}e^{-\xi S_\xi^-})b. \quad (3.11)$$

The first relation of (3.9) implies that $b = b(\xi S_\xi^-)$ and (3.11) then leads to:

$$b'(\xi S_\xi^-) \frac{1}{\xi} \text{sh} \xi S_\xi^- = b(\xi S_\xi^-) \frac{1}{\xi} \text{ch} \xi S_\xi^-$$

or $b = \frac{1}{\xi} \text{sh} \xi S_\xi^-$. We have used the relation

$$F(\xi S_\xi^-) S_\xi^0 = S_\xi^0 F(\xi S_\xi^-) + F'(\xi S_\xi^-) \frac{1}{\xi} \text{sh} \xi S_\xi^-$$

which follows from the deformed commutation relations and a particular case of which is (2.11). With this result for b the remaining relations (3.9) imply $a = S_\xi^0 + f(\xi S_\xi^-)$, $d = -S_\xi^0 + g(\xi S_\xi^-)$, $c = S_\xi^+ + x(g - f) + \xi h(\xi S_\xi^-)$, where f , g and h are arbitrary functions of ξS_ξ^- and x obeys the condition $[x, S_\xi^-] = -1$. The remaining conditions involved in (3.10) are

$$\begin{aligned} a(S_\xi^0 + \frac{1}{2}e^{\xi S_\xi^-}) - \xi b S_\xi^0 &= (S_\xi^0 + \frac{1}{2}e^{-\xi S_\xi^-})a, \\ d(S_\xi^0 - \frac{1}{2}e^{\xi S_\xi^-}) &= \xi S_\xi^0 b + (S_\xi^0 - \frac{1}{2}e^{-\xi S_\xi^-})d, \\ c(S_\xi^0 + \frac{1}{2}e^{\xi S_\xi^-}) - \xi d S_\xi^0 &= (S_\xi^0 - \frac{1}{2}e^{-\xi S_\xi^-})c + \xi S_\xi^0 a, \end{aligned}$$

and we have four further relations

$$a e^{\xi S_\xi^-} + b S_\xi^+ = S_\xi^+ b + e^{-\xi S_\xi^-} d,$$

$$\begin{aligned}
aS_\xi^+ - \xi bS_\xi^+ &= S_\xi^+ a + e^{-\xi S_\xi^-} c, \\
ce^{\xi S_\xi^-} + dS^+ &= \xi S_\xi^+ b + S_\xi^+ d, \\
cS_\xi^+ - \xi dS_\xi^+ &= S_\xi^+ c + \xi S_\xi^0 c,
\end{aligned}$$

which follow from the $a = +$ component of the symmetry relations (3.2). They are sufficient to obtain the explicit form of functions f , g and h and this leads to the result for the Lax operator,

$$\begin{aligned}
L^{(\xi)}(u) &= L^{(\xi)} + ul^{(\xi)}, \\
L^{(\xi)} &\equiv \begin{pmatrix} \frac{1}{2}\text{ch}\xi S_\xi^- + S_\xi^0 & \frac{1}{\xi}\text{sh}\xi S_\xi^- \\ S_\xi^+ & \frac{1}{2}\text{ch}\xi S_\xi^- - S_\xi^0 \end{pmatrix}, \quad l^{(\xi)} \equiv \begin{pmatrix} e^{-\xi S_\xi^-} & 0 \\ 2\xi S_\xi^0 + \xi\text{sh}\xi S_\xi^- & e^{\xi S_\xi^-} \end{pmatrix}.
\end{aligned} \tag{3.12}$$

The resulting Lax operator obeys QYBE in the form

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v) \tag{3.13}$$

due to the intertwining relation with the co-product symbols (2.16, 3.2), and this has been checked by direct but tedious calculations in the following way. We represent the fundamental R -matrix (3.3) and Lax operator (3.12) as follows:

$$\begin{aligned}
R_{ab}^{cd}(u) &= (\delta_1)_a^c (\delta_2)_b^d - \xi (\sigma_1^0)_a^b (\sigma_2^-)_b^d + \xi (\sigma_1^-)_a^b (\sigma_2^0)_b^d + \xi^2 (\sigma_1^-)_a^b (\sigma_2^-)_b^d, \\
(L_i(u))_a^b &= (u + \frac{1}{2})\text{ch}\xi S_\xi^- (\delta_i)_a^b + [S_\xi^0 - u\text{sh}\xi S_\xi^-] (\sigma_i^0)_a^b + \frac{1}{\xi}\text{sh}\xi S_\xi^- (\sigma^+)_a^b + [S_\xi^+ + u\xi(2S_\xi^0 + \text{sh}\xi S_\xi^-)] (\sigma_i^-)_a^b,
\end{aligned}$$

and multiplying Pauli matrices acting in the same space, we obtain on both sides of QYBE an expansion over terms of the type $\sigma_1^a \otimes \sigma_2^b$, $a, b = \pm, 0$ acting in $V_1^{-\frac{1}{2}} \otimes V_2^{-\frac{1}{2}}$ multiplied with operators acting in V_3^ℓ . Then we check that the operator-valued coefficients of the corresponding $\sigma_1^a \otimes \sigma_2^b$ term on both sides are equal by using the commutation relations (2.8) of the deformed generators.

We have the alternative option for constructing the Lax operator by the twist relation (2.15). It reads in the case at hand

$$L^{(\xi)}(u) = F_{\xi,21} L^{(0)}(u) F_{\xi,12}^{-1}, \tag{3.14}$$

with the well known undeformed Lax operator (with u shifted by $\frac{1}{2}$)

$$L^{(0)}(u) = \begin{pmatrix} u + \frac{1}{2} + S_0^0 & S_0^- \\ S_0^+ & u + \frac{1}{2} - S_0^0 \end{pmatrix}. \tag{3.15}$$

We write the twist element in the representation $\ell_1 = -\frac{1}{2}, \ell_2 = \ell$,

$$\begin{aligned}
F_{\xi,12} &= \exp(\frac{1}{2}\hat{\sigma}_1^0 \otimes \sigma_{\xi,2}), \quad e^{-\sigma_{\xi,2}} = 1 - 2\xi S_{0,2}^-, \\
F_{\xi,21} &= \exp(S_{0,2}^0 \otimes \hat{\sigma}_{\xi,1}), \quad e^{-\sigma_{\xi,1}} = I - 2\xi \hat{\sigma}^-.
\end{aligned} \tag{3.16}$$

Here Pauli matrices $\hat{\sigma}^a$ are to be distinguished from the two representations of the operator σ_ξ . The representation label ℓ on the operators acting in V^ℓ is suppressed and will be written as an additional subscript if necessary. Now the twist elements can also be written as 2×2 matrices with elements acting as operators in V^ℓ ,

$$F_{12} = \begin{pmatrix} e^{\sigma_\xi/2} & \\ & e^{-\sigma_\xi/2} \end{pmatrix} = \begin{pmatrix} (1 - 2\xi S_0^-)^{-\frac{1}{2}} & 0 \\ 0 & (1 - 2\xi S_0^-)^{\frac{1}{2}} \end{pmatrix},$$

$$F_{21} = \begin{pmatrix} 1 & 0 \\ 2\xi S_0^0 & 1 \end{pmatrix}. \quad (3.17)$$

As the result we obtain the Lax operator in terms of the undeformed generators S_0^a in the generic representation ℓ ,

$$L^{(\xi)}(u) = \begin{pmatrix} (u + \frac{1}{2} + S_0^0)(1 - 2\xi S_0^-)^{\frac{1}{2}} & S_0^-(1 - 2\xi S_0^-)^{-\frac{1}{2}} \\ [(u + \frac{1}{2} + S_0^0)2\xi S_0^0 + S^+](1 - 2\xi S_0^-)^{\frac{1}{2}} & [2\xi S_0^0 S_0^- + u + \frac{1}{2} - S_0^0](1 - 2\xi S_0^-)^{-\frac{1}{2}} \end{pmatrix}. \quad (3.18)$$

The comparison of both results (3.12, 3.18) leads to the following expressions of the deformed generators S_ξ^a in terms of the undeformed ones S_0^a ,

$$\begin{aligned} S_\xi^- &= -\frac{1}{2\xi} \ln(1 - 2\xi S_0^-), \\ S_\xi^0 &= S_0^0(1 - 2\xi S_0^-)^{\frac{1}{2}} - \frac{\xi}{2} S_0^-(1 - 2\xi S_0^-)^{-\frac{1}{2}}, \\ S_\xi^+ &= [S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})](1 - 2\xi S_0^-)^{\frac{1}{2}}. \end{aligned} \quad (3.19)$$

With these expressions one can prove by direct computation the commutation relations (2.8) assuming that S_0^a obey the undeformed $sl(2)$ algebra relations (2.3). This calculation is outlined in Appendix A.

There is another relation of the deformed generators S_ξ^a to the undeformed $sl(2)$ algebra [29]. The three functions of S_ξ^a ,

$$S_J^- = \frac{\text{sh}(\xi S_\xi^-)}{\xi(\text{ch}(\frac{\xi}{2} S_\xi^-))^2}, \quad S_J^0 = S_\xi^0, \quad S_J^+ = \text{ch}(\frac{\xi}{2} S_\xi^-) S_\xi^+ \text{ch}(\frac{\xi}{2} S_\xi^-), \quad (3.20)$$

obey the undeformed commutation relations (2.3) (with S_0^a replaced by S_J^a).

3.3 Universal R operator

We consider QYBE for generic ℓ_1 and ℓ_2 and $\ell_3 = -\frac{1}{2}$. It involves the universal R operator, $R^{(\xi, \ell_1, \ell_2)}(u)_{12}$ and $R_{i3}^{(\xi, \ell_i, -\frac{1}{2})}$, $i = 1, 2$ which can be represented by the Lax matrices just obtained (3.12),

$$R_{12}^{(\xi)}(u) L_1^{(\xi)}(u+v) L_2^{(\xi)}(v) = L_2^{(\xi)}(v) L_1^{(\xi)}(u+v) R_{12}^{(\xi)}(u). \quad (3.21)$$

The Lax operators involve the generators acting on space 1 or 2 correspondingly in the representations ℓ_1 or ℓ_2 . This 2×2 matrix relation serves as the defining condition for the universal R operator acting on $V_1^{\ell_1} \otimes V_2^{\ell_2}$. By decomposition in powers of v we obtain three conditions. The first two,

$$R_{12}(u) l_1 l_2 = l_2 l_1 R_{12}(u), \quad (3.22)$$

$$R_{12}(u)(L_1 l_2 + l_1 L_2) = (L_2 l_1 + l_2 L_1) R_{12}(u),$$

are equivalent to the symmetry relations (3.2), now for generic representations. The third condition reads

$$R_{12}(u) K_L(u) = K_R(u) R_{12}(u),$$

$$\begin{aligned} K_L(u) &= L_1 L_2 + \frac{u}{2}(l_1 L_2 - L_1 l_2), \\ K_R(u) &= L_2 L_1 + \frac{u}{2}(L_2 l_1 - l_2 L_1). \end{aligned} \quad (3.23)$$

The elements of the 2×2 matrices $K_{L/R}$ are operators on $V_1^{\ell_1} \otimes V_2^{\ell_2}$ and their explicit form is listed in Appendix B.,

Together with the symmetry conditions (3.2) one of the four conditions involved in the 2×2 matrix relation (3.23) is sufficient to fix the operator $R_{12}^{(\xi)}(u)$ up to normalization.

From the deformed commutation relations (2.8)) we see that S_ξ^a may be represented in a form even in the deformation variable ξ . The dependence on ξ is not on even powers only for the form (3.19) with the S_0^a independent on ξ . For S_ξ^a symmetric in ξ the second term in $L^\xi(u) = L^{(\xi)} + ul^{(\xi)}$ (3.12) is not symmetric with respect to $\xi \leftrightarrow -\xi$. Because

$$(L^{(\xi)})^2 = (C_\xi + \frac{1}{4})I, \quad l^{(-\xi)} = (l^{(\xi)})^{-1}, \quad l^{(\xi)}L^{(\xi)} = L^{(\xi)}l^{(-\xi)}, \quad (3.24)$$

we find that

$$L^{(-\xi)}(-u) = (C_\xi + \frac{1}{4} - u^2) (L^{(\xi)}(u))^{-1}, \quad (3.25)$$

i.e. on a given irreducible representation the inverse of $L^{(\xi)}(u)$ is proportional to $L^{(-\xi)}(-u)$. This property holds also for the universal R operator $R^{(\xi, \ell_1, \ell_2)}$ in accordance with the two forms of QYBE, the above one (3.21) and the following one,

$$R_{21}^{(-\xi)}(u) L_2^{(-\xi)}(u+v)_1 L_1^{(-\xi)}(v) = L_2^{(-\xi)}(v) L_2^{(-\xi)}(u+v) R_{12}^{(-\xi)}(u). \quad (3.26)$$

We notice that the Lax operator can be made symmetric in ξ (for symmetric S_ξ^a) by the following similarity transformation,

$$L^{(\xi)sym}(u) = e^{u\xi S_{\xi,1}^-} L^{(\xi)}(u) e^{-u\xi S_{\xi,1}^-} = \begin{pmatrix} \frac{1}{2}\text{ch}\xi S_\xi^- + S_\xi^0 & \frac{1}{\xi}\text{sh}\xi S_\xi^- \\ S_\xi^+ & \frac{1}{2}\text{ch}\xi S_\xi^- - S_\xi^0 \end{pmatrix} + u \begin{pmatrix} \text{ch}\xi S_\xi^- & 0 \\ (u+1)\xi\text{sh}\xi S_\xi^- & \text{ch}\xi S_\xi^- \end{pmatrix}. \quad (3.27)$$

The resulting matrix operator obeys QYBE with the fundamental representation R matrix (3.4) as well as the original one. Taking into account that the universal R operator commutes with $S_{\xi,1}^- + S_{\xi,2}^-$ we obtain that

$$R_{12}^{(\xi)sym}(u) = e^{u\xi S_\xi^-} R_{12}^{(\xi)}(u) e^{-u\xi S_\xi^-}, \quad (3.28)$$

obeys (3.21) with $L^{(\xi)}$ replaced by $L^{(\xi)sym}$ and is therefore symmetric in $\xi \leftrightarrow -\xi$.

4 Representations by polynomials of one variable

Guided by the motivations described in the Introduction we would like to represent the considered algebras in terms of operators acting on polynomial functions. We consider the representations with one variable in this section and the tensor product representations involving two variables in the next section.

4.1 Representations of the deformed algebra

Recall that the undeformed generators (2.3) can be represented on functions $\phi(y)$ as differential operators,

$$S_{0,\ell}^- = \partial_y, \quad S_{0,\ell}^0 = \frac{1}{2}(y\partial_y + \partial_y y) + \ell - \frac{1}{2}, \quad S_{0,\ell}^+ = -y\partial_y y - (2\ell - 1)y. \quad (4.1)$$

Representations with conformal weight ℓ are spanned by the polynomials

$$\phi_\ell^{(m)}(y) = (S_{0,\ell}^+)^m \phi_{0,\ell}^{(0)}(y), \quad (4.2)$$

with the lowest weight state represented by $\phi_{0,\ell}^{(0)}(y) = 1$. Up to normalization this basis is given by the monomials, $\phi_\ell^{(m)}(y) = y^m$. In the particular case $\ell = 0$ $S_{0,0}^a$ generate the *Möbius* transformations of the variable y ,

$$y \rightarrow y' = \frac{ay + b}{cy + d}.$$

The deformed generators (2.8) can be represented as

$$S_{\xi,\ell}^- = \partial_x, \quad S_{0,\ell}^0 = \frac{1}{2}(x\partial_x^\xi + \partial_x^\xi x) + \ell - \frac{1}{2}, \quad S_{\xi,\ell}^+ = -x\partial_x^\xi x - (2\ell - 1)x, \quad (4.3)$$

where

$$\partial_x^\xi = \frac{1}{\xi} \text{sh}(\xi \partial_x).$$

Substituting these generators into the expression of the Casimir operator (2.10) we confirm that the restriction of C_ξ to the representation with weight ℓ is the identity operator times $\ell(\ell - 1)$.

If we would substitute the representations (4.1) for $S_{0,\ell}^a$ to (3.19) we would obtain a quite different representation of the deformed generators in terms of (y, ∂_y) . That representation is more complicated. First of all it has the disadvantage that $S_{\xi,\ell}^-$ does not appear as the infinitesimal translation in y , further, it is not symmetric in ξ .

The lowest weight states of the deformed representation ℓ obey

$$S_{\xi,\ell}^- \varphi_\ell^{(0)}(x) = 0, \quad S_{\xi,\ell}^0 \varphi_\ell^{(0)}(x) = \ell \varphi_\ell^{(0)}(x), \quad (4.4)$$

and they are again represented by the constant functions, $\varphi_\ell^{(0)}(x) = 1$. The basis polynomials of this representation obeying the eigenvalue condition

$$S_{\xi,\ell}^0 \varphi_\ell^{(m)} = (\ell + m) \varphi_\ell^{(m)} \quad (4.5)$$

cannot be obtained by applying $S_{\xi,\ell}^+$ to the constant. In order to construct these basis polynomials we consider the *van der Jeugt* operators S_J^a (3.20). Using the representation (4.3) we can write them in terms of the following *Heisenberg* canonical pair,

$$X_J = \text{ch}\left(\frac{\xi}{2}\partial_x\right)x\text{ch}\left(\frac{\xi}{2}\partial_x\right), \quad D_J = (\text{ch}\left(\frac{\xi}{2}\partial_x\right)^{-2}\partial_x^\xi, \quad [D_J, X_J] = 1, \quad (4.6)$$

in the form

$$S_{J,\ell}^- = D_J, \quad S_{J,\ell}^0 = \frac{1}{2}(X_J D_J + D_J X_J) + \ell - \frac{1}{2}, \quad S_{J,\ell}^+ = -X_J D_J X_J - (2\ell - 1)X_J. \quad (4.7)$$

$\phi_\ell^{(m)}(X_J)$ span the representation ℓ of the undeformed algebra generated by $S_{J,\ell}^a$. On the other hand, considering the generators $S_{J,\ell}^a$ as operators on polynomial functions of x , the conditions (4.4, 4.5) are equivalent to

$$S_{J,\ell}^- \varphi_\ell^{(0)}(x) = 0, \quad S_{J,\ell}^0 \varphi_\ell^{(m)}(x) = (\ell + m) \varphi_\ell^{(m)}(x), \quad (4.8)$$

We obtain for the basis polynomials of the deformed representation

$$\varphi_\ell^{(m)}(x) = (S_{J,\ell}^+)^m \cdot 1 = \text{const } X_J^m \cdot 1. \quad (4.9)$$

The explicit polynomials are obtained by substituting X_J in terms of (x, ∂_x) (4.6) and commuting the differential operators to the right. Specifying the normalization to be such that the coefficient of the highest power is 1 we have e.g.

$$\begin{aligned} \varphi^{(0)} &= 1, \quad \varphi^{(1)} = x, \quad \varphi^{(2)} = x^2 + \frac{1}{4}, \quad \varphi^{(3)} = x^3 + \frac{5}{4}x, \quad \varphi^{(4)} = x^4 + \frac{7}{2}x^2 + \frac{9}{16}, \\ \varphi^{(5)} &= x^5 + \frac{15}{2}x^3 + \frac{89}{16}x, \quad \varphi^{(6)} = x^6 + \frac{55}{4}x^4 + \frac{439}{16}x^2 + \frac{225}{64}. \end{aligned} \quad (4.10)$$

The deformed basis polynomials $\varphi_\ell^{(m)}$ can be computed by the following rule: Expand

$$[(2m_1 + 1)!!]^2 \left(\frac{\xi^2}{4} + p \right)^{m_1}$$

in powers of p and substitute for even $m = 2m_1 + 2$,

$$p^k \rightarrow x \prod_{r=-k+1}^{k-1} (x + r\xi)[(2k-1)!!]^{-2},$$

and for odd $m = 2m_1 + 1$

$$p^k \rightarrow \prod_{r=-k}^k (x + r\xi)[(2k+1)!!]^{-2} \quad (4.11)$$

The derivation is given in Appendix C.

There is also a generating function for these polynomials

$$G_1(x, t) \equiv \sum_{m=0}^{\infty} \frac{t^m}{m!} \varphi^{(m)}(x) = \frac{\left(1 + \frac{t\xi}{2}\right)^{\frac{x}{\xi} - \frac{1}{2}}}{\left(1 - \frac{t\xi}{2}\right)^{\frac{x}{\xi} + \frac{1}{2}}}. \quad (4.12)$$

For a derivation we refer to Appendix C. It can be checked that the generating function $G_1(x, t)$ satisfies the following difference-differential equation,

$$\frac{x}{2\xi} (G_1(x + \xi, t) - G_1(x - \xi, t)) + \frac{1}{4} (G_1(x + \xi, t) + G_1(x - \xi, t) - 2G_1(x, t)) = t \frac{\partial}{\partial t} G_1(x, t),$$

which is the implication of the eigenvalue condition in (4.8) read as a difference equation for $\varphi_\ell^{(m)}$.

4.2 Induced representation

The preferred representation of the deformed generators (4.3), where $S_\xi^- = \partial_x$ is the infinitesimal translation, is connected via the twist (3.19) to a representation of the undeformed generators S_0^a .

We use a two-parameter family of *Heisenberg* canonical pairs,

$$D_x^{(\xi)} = e^{-\xi\partial_x} \partial_x^\xi, \quad X^{\xi,\alpha} = (1 + e^{-\xi\partial_x})^{-\alpha} e^{\xi\partial_x} x e^{\xi\partial_x} (1 + e^{-\xi\partial_x})^\alpha, \quad [D_x^{(\xi)}, X^{\xi,\alpha}] = 1, \quad (4.13)$$

and show that the representation of $S_{0,\ell}^a$ induced by the one of $S_{\xi,\ell}^a$ (with the $a = -$ component generating translations in x) is given by (4.1) with the pair (y, ∂_y) replaced by the pair $(X^{(\xi,2\ell-1)}, D_x^{(\xi)})$.

Indeed, inverting the first relation of (3.19) we have

$$S_{0,\ell}^- = \frac{1}{2\xi} \left(1 - e^{-2\xi S_{\xi,\ell}^-} \right) = D_x^{(\xi)}, \quad \sqrt{1 - 2\xi S_{0,\ell}^-} = e^{-\xi\partial_x} \quad (4.14)$$

The second relation in (3.19) can be written as

$$S_{0,\ell}^0 = \left(S_{\xi,\ell}^0 + \frac{\xi}{2} D_x^{(\xi)} e^{\xi\partial_x} \right) e^{\xi\partial_x}.$$

Substituting the representation (4.3) we obtain as an intermediate form

$$S_{0,\ell}^0 = \frac{1}{2} \left(X^{(\xi,0)} D_x^{(\xi)} + D_x^{(\xi)} X^{(\xi,0)} \right) + \left(\ell - \frac{1}{2} \right) e^{\xi\partial_x}.$$

Now we substitute

$$X^{(\xi,0)} = X^{(\xi,\alpha)} - \frac{\alpha \xi e^{2\xi\partial_x}}{1 + e^{\xi\partial_x}}, \quad (4.15)$$

and obtain for $\alpha = 2\ell - 1$

$$S_{0,\ell}^0 = \frac{1}{2} \left(X^{(\xi,2\ell-1)} D_x^{(\xi)} + D_x^{(\xi)} X^{(\xi,2\ell-1)} \right) + \ell - \frac{1}{2}. \quad (4.16)$$

The third relation in (3.19) leads to

$$S_{0,\ell}^+ = S_{\xi,\ell}^+ e^{\xi\partial_x} - 2\xi \left[\frac{1}{2} \left(X^{(\xi,2\ell-1)} D_x^{(\xi)} + D_x^{(\xi)} X^{(\xi,2\ell-1)} \right) + \left(\ell - \frac{1}{2} \right) \right] \cdot \left[\frac{1}{2} \left(X^{(\xi,2\ell-1)} D_x^{(\xi)} + D_x^{(\xi)} X^{(\xi,2\ell-1)} \right) + \ell \right].$$

We substitute (4.3) and show that the remainder in

$$S_{0,\ell}^+ = -X^{(\xi,2\ell-1)} D_x^{(\xi)} X^{(\xi,2\ell-1)} - (2\ell - 1) X^{(\xi,2\ell-1)} + (S_{0,\ell}^+)_r \quad (4.17)$$

vanishes, $(S_{0,\ell}^+)_r = 0$.

The undeformed generators are represented now by rather involved difference operators in (x, ∂_x) and they depend now on the deformation parameter ξ . The basis polynomials of the representation ℓ of the undeformed algebra, being simple monomials in y , become now more involved polynomials in x ,

$$\phi_{x,\ell}^{(m)}(x) = \phi_{x,\ell}^{(m)}(X^{(\xi,2\ell-1)}) \cdot 1 = 2^{-(2\ell-1)m} \prod_{k=1}^m (x + (\ell - \frac{1}{2} + 2k - 1)\xi). \quad (4.18)$$

The original form of the undeformed representations (4.1) will be referred to as the y picture whereas the induced form (4.14, 4.16, 4.17) will be called x picture.

5 Polynomial representations of tensor products

5.1 Undeformed tensor product

For comparison we recall the undeformed case, where $\Delta_0(S_0^a) = S_{0,1}^a + S_{0,2}^a = S_{0,12}^a$ and where the lowest weight states of the irreducible representations appearing in the tensor product of representations ℓ_1 and ℓ_2 are represented by the solutions of

$$S_{0,12}^- \phi_{12,\ell_1,\ell_2,n}^{(0)} = 0, \quad S_{0,12}^0 \phi_{12,\ell_1,\ell_2,n}^{(0)} = (\ell_1 + \ell_2 + n) \phi_{12,\ell_1,\ell_2,n}^{(0)}. \quad (5.1)$$

We shall usually suppress the labels ℓ_1, ℓ_2 .

In the (y, ∂_y) picture (4.1) we see, that $\phi_{12,n}^{(0)}$ depends on the difference $y_1 - y_2 = y_{12}$ only and that

$$\phi_{12,n}^{(0)}(y_1, y_2) = y_{12}^n. \quad (5.2)$$

The states of an irreducible representation with weight $\ell = \ell_1 + \ell_2 + n$ are generated as

$$\phi_{12,n}^{(m)}(y_1, y_2) = (S_{0,12}^+)^m \phi_{12,n}^{(0)}(y_1, y_2). \quad (5.3)$$

The same undeformed representation can also be described in the (x, ∂_x) picture (4.14, 4.16, 4.17). The polynomial $\phi_{x,12,n}^{(m)}(x_1, x_2)$ is obtained from the above one by substituting in the decomposition of (5.3) in the monomials $y_1^{m_1} y_2^{m_2}$, according to (4.18),

$$y_i^{m_i} \rightarrow 2^{-(2\ell_i-1)m_i} \prod_{k=1}^{m_i} (x_i + (\ell_i - \frac{1}{2} + 2k - 1)\xi). \quad (5.4)$$

In particular the lowest weight polynomials do not depend on the difference x_{12} only.

5.2 Deformed tensor product

The generators $S_{\xi,12}^a$ acting on $V_1^{\ell_1} \otimes V_2^{\ell_2}$ are given by (2.9, 4.3) and the conditions on the polynomials representing the lowest weight states of the irreducible representations appearing in the deformed tensor product are

$$S_{\xi,12}^- \varphi_{12,\ell_1,\ell_2,n}^{(0)} = 0, \quad S_{\xi,12}^0 \varphi_{12,\ell_1,\ell_2,n}^{(0)} = (\ell_1 + \ell_2 + n) \varphi_{12,\ell_1,\ell_2,n}^{(0)}. \quad (5.5)$$

The labels ℓ_1, ℓ_2 will be suppressed. Since $S_{\xi,12}^- = \partial_{x_1} + \partial_{x_2}$ we see that here again the dependence is on the difference $x_1 - x_2 = x_{12}$ only. Using this fact the second condition can be written in the form

$$[\partial_{x_1}^\xi x_{12} + \ell_1 + \ell_2 - 1] e^{-\xi \partial_{x_1}} \varphi_{12,n}^{(0)}(x_{12}) = (\ell_1 + \ell_2 + n) \varphi_{12,n}^{(0)}(x_{12}).$$

We shall solve this eigenvalue equation by expressing the involved operator in terms of the canonical *Heisenberg* pair (4.13). We abbreviate x_{12} by x and then the latter equation can be written as

$$[D_x^{(-\xi)} X^{(-\xi,0)} + (e^{-\xi \partial_x} - 1)(\ell_1 + \ell_2 - 1)] \varphi_{12,n}^{(0)} = (n + 1) \varphi_{12,n}^{(0)}.$$

We notice that

$$e^{\xi \partial_x} - 1 = -2\xi D_x^{(-\xi)} \frac{e^{-2\xi \partial_x}}{1 + e^{\xi \partial_x}}$$

and use also (4.15) with ξ replaced by $-\xi$ and α by $2(\ell_1 + \ell_2 - 1)$ to obtain the eigenvalue equation in the form

$$D_x^{(-\xi)} X^{(-\xi, 2(\ell_1 + \ell_2 - 1))} \varphi_{12,n}^{(0)} = (n+1) \varphi_{12,n}^{(0)} \quad (5.6)$$

In the $(X^{(-\xi, 2(\ell_1 + \ell_2 - 1))}, D^{(-\xi)})$ picture the eigenfunctions are just

$$\tilde{\varphi}_{12,n}^{(0)} = (X^{(-\xi, 2(\ell_1 + \ell_2 - 1))})^n$$

with n all non-negative integers. Returning to the (x, ∂_x) picture we obtain the polynomials representing the lowest weight states in the ξ deformed tensor product of representations with conformal weights ℓ_1, ℓ_2 as

$$\begin{aligned} \varphi_{12,n}^{(0)} &= (X^{(-\xi, 2(\ell_1 + \ell_2 - 1))})^n \cdot 1|_{x=x_{12}} = \\ &2^{-(2(\ell_1 + \ell_2 - 1)n)} \prod_{k=1}^n (x_{12} - \xi(\ell_1 + \ell_2 - 1 + 2k - 1)). \end{aligned} \quad (5.7)$$

Actually we have just solved a non-trivial difference equation. The eigenvalue condition (5.5) with $S_{\xi,12}^0$ acting on functions of $x = x_{12}$ reads

$$\begin{aligned} \frac{x}{2\xi} [\varphi_n(x) - \varphi_n(x - 2\xi)] + (\ell_1 + \ell_2) \varphi_n(x - \xi) + \frac{1}{2} [\varphi_n(x) + \varphi_n(x - 2\xi) - 2\varphi_n(x - \xi)] &= \\ &= (\ell_1 + \ell_2 + n) \varphi_n(x), \end{aligned} \quad (5.8)$$

We shall denote the left-hand side by $\tilde{S}_{\xi,x}^0 \varphi_n$.

By representing the function by a *Fourier* integral one can deduce similar to the one-point case (8.12) an expression of these polynomials

$$\varphi_n(x) = c_n \int dk e^{ik(\frac{x}{\xi} + 1 - \ell_1 - \ell_2 - n)} (\sin k)^{-\ell_1 - \ell_2 - n} \left(\frac{\cos \frac{k}{2}}{\sin \frac{k}{2}} \right)^{1 - \ell_1 - \ell_2}.$$

and derive from it recursive relations like

$$(1 - n - \ell_1 - \ell_2) \varphi_n(x) - (1 - \ell_1 - \ell_2) \varphi_n(x - \xi) = \frac{c_n}{c_{n-1}} i \left(1 - \frac{x}{\xi}\right) \varphi_{n-1}(x - \xi),$$

and

$$\varphi_n(x) - \varphi_n(x - 2\xi) = 2i \frac{c_n}{c_{n-1}} \varphi_{n-1}(x - 2\xi). \quad (5.9)$$

The explicite expression obtained above (5.7) obeys indeed these relations with the coefficients related as

$$c_n = -in c_{n-1}.$$

The difference equation (5.8) can be solved directly, starting from the special case $\ell_1 + \ell_2 = 1$ where it simplifies to

$$\left(\frac{x}{\xi} - 1\right) \frac{1}{2} [\varphi_n(x) - \varphi_n(x - 2\xi)] = n \varphi_n(x).$$

and leads to the solution (5.7) for this particular case. We summarize this solution for all n into the generating function

$$G_2^{(+1)}(x, \tau) = \sum_{n=0}^{\infty} \tau^n \varphi_n(x) = (1 + 2\tau\xi)^{\frac{x-\xi}{2\xi}}.$$

The equation (5.5) in the general case is obtained from this special one by similarity transformation with $(1 + e^{\xi \partial_x})^{-2(\ell_1 + \ell_2 - 1)}$. This allows to recover the above result (5.7) and also to represent the general case lowest weight polynomials by the following generating function

$$G_2^{(\ell_1 + \ell_2)}(x, \tau) = (1 + e^{\xi \partial_x})^{-2(\ell_1 + \ell_2 - 1)} (1 + 2\tau\xi)^{\frac{x - \xi}{2\xi}} = (1 + 2\tau\xi)^{\frac{x - \xi}{2\xi}} (1 + \sqrt{1 + 2\tau\xi})^{-2(\ell_1 + \ell_2 - 1)}. \quad (5.10)$$

6 The universal R operator

6.1 The spectral form

According to (2.16) the universal R operator $R_{12, \ell_1, \ell_2}^{(\xi)}(u)$ maps the irreducible representation $\ell = \ell_1 + \ell_2 + n$ in the tensor product $V_1^{\ell_1} \otimes V_2^{\ell_2}$ constructed according to Δ_ξ into the corresponding one of $\bar{\Delta}_\xi = \Delta_{-\xi}$ and vice versa. In particular the symmetry relations (3.2) guarantee that the basis functions are mapped into each other. This is expressed by the following generalized eigenvalue relations,

$$\begin{aligned} R_{12, \ell_1, \ell_2}^{(\xi)}(u) \varphi_{12, n}^{(\xi), (m)}(x_1, x_2) &= \rho_{\ell_1, \ell_2, n} \varphi_{12, n}^{(-\xi), (m)}(x_1, x_2), \\ R_{12, \ell_1, \ell_2}^{(\xi)}(u) \varphi_{12, n}^{(-\xi), (m)}(x_1, x_2) &= \bar{\rho}_{\ell_1, \ell_2, n} \varphi_{12, n}^{(\xi), (m)}(x_1, x_2). \end{aligned} \quad (6.1)$$

By symmetry the eigenvalues $\rho_{\ell_1, \ell_2, n}$ do not depend on the level m . We have pointed out above that $R_{12}^{(\xi)}$ can be made symmetric in $\xi \leftrightarrow -\xi$ by a similarity transformation. This implies that the eigenvalues do not depend on the sign of ξ , therefore

$$\rho_{\ell_1, \ell_2, n} = \bar{\rho}_{\ell_1, \ell_2, n}.$$

We check now that the above eigenvalue relations are compatible with the defining conditions of the universal R operator if applied to the lowest weight basis polynomials ($m = 0$). It is sufficient to consider the conditions (3.23) restricted to the upper right (12) element in the 2×2 matrices,

$$R_{12, \ell_1, \ell_2}^{(\xi)}(u) K_L^{12}(u) \varphi_{12, n}^{(-\xi), (m)}(x_1, x_2) = K_R^{12}(u) R_{12, \ell_1, \ell_2}^{(\xi)}(u) \varphi_{12, n}^{(-\xi), (m)}(x_1, x_2) \quad (6.2)$$

We consider the action of $K_{L/R}^{12}(u)$ as difference operators on functions of $x_{12} = x$,

$$\begin{aligned} 2\xi K_{L/R} \varphi(x) &= \pm \frac{x}{2\xi} [2\varphi(x) - \varphi(x + 2\xi) - \varphi(x - 2\xi)] \\ &\quad \pm (1 - \ell_1 - \ell_2) [\varphi(x + \xi) - \varphi(x - \xi)] \\ &\quad \mp \frac{1}{2} [\varphi(x + 2\xi) - \varphi(x - 2\xi)] \pm u [\varphi(x \mp 2\xi) - \varphi(x)] \end{aligned} \quad (6.3)$$

We compare with the action of the generator $S_{\xi, 12}^0$ as a difference operator on functions of $x_{12} = x$ (5.8) and use the notation $\tilde{S}_{\pm\xi, x}^0 \varphi(x)$ introduced there,

$$2\xi K_{L/R}^{12}(u) \varphi(x) = (\tilde{S}_{\mp\xi, x}^0 \pm u) (\varphi(x \mp 2\xi) - \varphi(x)). \quad (6.4)$$

We know that $\varphi_{12, n}^{(\pm\xi), (0)}(x)$ are eigenfunctions of $\tilde{S}_{\pm\xi, 12}^0$ with eigenvalue $\ell_1 + \ell_2 + n$. We use also the iterative relation (5.9) to obtain

$$2\xi K_{L/R}^{12}(u) \varphi_{12, n}^{(\mp\xi), (0)}(x) = \mp 2n (\ell_1 + \ell_2 + n \pm u) \varphi_{12, n}^{(\mp\xi), (0)}(x). \quad (6.5)$$

We use now the latter result and the eigenvalue relations (6.1) to evaluate both sides of (6.2) and obtain

$$-(\ell_1 + \ell_2 + n + u)\rho_{\ell_1, \ell_2, n-1} \varphi_{12, n-1}^{(-\xi), (0)}(x) = (\ell_1 + \ell_2 + n - u)\rho_{\ell_1, \ell_2, n} \varphi_{12, n}^{(-\xi), (0)}(x). \quad (6.6)$$

This proves the compatibility of the eigenvalue relations with the defining conditions of the universal R operator and results in the iterative relation for the eigenvalues with the solution

$$\rho_{\ell_1, \ell_2, n} = \text{const}(-1)^n \frac{\Gamma(\ell_1 + \ell_2 + n + u + 1)\Gamma(\ell_1 + \ell_2 - u)}{\Gamma(\ell_1 + \ell_2 + u)\Gamma(\ell_1 + \ell_2 + n + 1 - u)}. \quad (6.7)$$

The eigenvalues do not depend on the deformation parameter ξ and coincide with the undeformed ones. This is expected from the twist relation between the deformed and undeformed universal R operators to be considered now. We compare the eigenvalue relations (6.1) with the one for the undeformed universal R operator,

$$R_{12, \ell_1, \ell_2}^{(0)}(u) \phi_{12, n}^{(m)} = \rho_{\ell_1, \ell_2, n}^0 \phi_{12, n}^{(m)}. \quad (6.8)$$

We specify the twist relation (2.15) to the considered generic representations $V_1^{\ell_1} \otimes V_2^{\ell_2}$,

$$R_{12, \ell_1, \ell_2}^{(\xi)}(u) = F_{\xi, 21}^{\ell_2, \ell_1} R_{12, \ell_1, \ell_2}^{(0)}(u) (F_{\xi, 12}^{\ell_1, \ell_2})^{-1} \quad (6.9)$$

writing the twist element (2.17) in the appropriate representations explicitly as operators on functions of x_1, x_2 ,

$$F_{\xi, 12}^{\ell_1, \ell_2} = \exp(S_{0,1}^0 \sigma_{\xi, 2}) = \exp(\xi S_{0, x_1}^0 \partial_{x_2}), \quad (6.10)$$

where

$$\sigma_{\xi, 2} = -\ln(1 - 2\xi S_{0, x_2}^-) = \xi \partial_{x_2},$$

$$S_{0, x_1}^0 = \frac{1}{2} \left(X_1^{(-\xi, 2\ell_1-1)} D_{x_1}^{(-\xi)} + D_{x_1}^{(-\xi)} X_1^{(-\xi, 2\ell_1-1)} \right) + \ell_1 - \frac{1}{2}.$$

The comparison of the eigenvalue relations (6.1, 6.8) with respect to the twist relation implies the non-deformation of the eigenvalues, $\rho_{\ell_1, \ell_2, n} = \rho_{\ell_1, \ell_2, n}^0$, and the following relation of the basis polynomials of the deformed and induced undeformed representations,

$$\phi_{12, n}^{(m)}(x_1, x_2) = (F_{\xi, 12}^{\ell_1, \ell_2})^{-1} \varphi_{12, n}^{(\xi), (m)}(x_1, x_2) =: \varphi_{12, n}^{(\xi), (m)}(x_1, x_2 - S_{0, x_1}^0) : \cdot 1 \quad (6.11)$$

The last expression is to be understood from the expansion of the polynomial in powers of x_1 and x_2 , ordering in each term the operators S_{0, x_1}^0 to the left from any power of x_1 and acting with the resulting operator on the constant 1.

Above we have given an explicit expression for the polynomials representing the highest weights in the deformed tensor product (5.7) and a rule for the computation of the basis polynomials of the undeformed tensor product in the induced picture (5.4). The relation between these series of polynomials is quite non-trivial; a direct check has been done for some lowest order polynomials only.

6.2 The integral form at small ξ

In the polynomial representation considered the universal R operator acts on functions of two variables. We would like to represent this action in integral form,

$$R_{12,\ell_1,\ell_2}^{(\xi)}(u) \varphi(x_1, x_2) = \int dx_1' dx_2' \mathcal{R}_{\ell_1,\ell_2}^{(\xi)}(u; x_1, x_2; x_1', x_2') \varphi(x_1', x_2'). \quad (6.12)$$

We assume that both integrations go along closed contours in order to have simple integrations by part.

QYBE in the representation of generic ℓ_1, ℓ_2 and $\ell_3 = -\frac{1}{2}$ which serves as the defining conditions for the universal R operator with the known Lax operators implies now conditions on the kernel \mathcal{R} . We specify to the form with symmetry in $\xi \leftrightarrow -\xi$ and write down the Lax matrix (3.27)

$$\begin{aligned} L_{i,\ell_i}^{(\xi)sym}(u) &= L_{i,\ell_i}^{(\xi)sym} + u(\text{ch}(\xi\partial_i)I + (u+1)\hat{\sigma}^- \text{sh}(\xi\partial_i)), \\ L_{i,\ell_i}^{(\xi)sym} &= \frac{1}{2}\text{ch}(\xi S_{\xi,i,\ell_i}^-)I + \begin{pmatrix} S_{\xi,i,\ell_i}^0 & \frac{1}{\xi}\text{sh}(\xi S_{\xi,i,\ell_i}^-) \\ S_{\xi,i,\ell_i}^+ & -S_{\xi,i}^0 \end{pmatrix}, \end{aligned} \quad (6.13)$$

and also its transposition, appearing by partial integration with respect to x_i ,

$$(L_i^{(\xi)sym}(u))^T = -L_{i,\ell_i^T}^{(\xi)sym} + (u+1)(\text{ch}(\xi\partial_i)I - u\hat{\sigma}^- \text{sh}(\xi\partial_i)), \quad (6.14)$$

Notice that T does not transpose the 2×2 matrix and that $\ell_i^T = 1 - \ell_i$. We keep in mind that the generators in L_{i,ℓ_i} are operators acting on x_i in the representation ℓ_i . We suppress the representation label adopting the convention $\ell_{i'} = 1 - \ell_i, 1 = 1, 2$. In these notations the defining condition for the kernel reads,

$$\begin{aligned} & \left[L_2^{(\xi)sym} + v(\text{ch}(\xi\partial_2)I + (v+1)\hat{\sigma}^- \text{sh}(\xi\partial_2)) \right] \\ & [L_1^{(\xi)sym} + (u+v)(\text{ch}(\xi\partial_1)I + (u+v+1)\hat{\sigma}^- \text{sh}(\xi\partial_1))] \\ & - [L_{1'}^{(\xi)sym} - (u+v+1)(\text{ch}(\xi\partial_{1'})I - (u+v)\hat{\sigma}^- \text{sh}(\xi\partial_{1'}))] \\ & [L_{2'}^{(\xi)sym} - (v+1)(\text{ch}(\xi\partial_{2'})I - v\hat{\sigma}^- \text{sh}(\xi\partial_{2'}))] \Big] \\ & \mathcal{R}_{\ell_1,\ell_2}^{(\xi)}(u; x_1, x_2; x_1', x_2') = 0 \end{aligned} \quad (6.15)$$

From this matrix condition we obtain two conditions projecting

$$\begin{pmatrix} \hat{A}\mathcal{R} & \hat{B}\mathcal{R} \\ \hat{C}\mathcal{R} & \hat{D}\mathcal{R} \end{pmatrix} = 0$$

by multiplication from the left by the row $(x_l, 1)$ and from the right by the column $(1, -x_r)^T$. In the first case we put $x_l = x_2$ and $x_r = x_{2'}$, in the second case $x_l = x_1$ and $x_r = x_{1'}$ after the differential operations are done, i.e. we understand that the operators $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ act on \mathcal{R} only. In this way we obtain the defining relations for the kernel of the universal R operator

$$(x_2\hat{A} + \hat{C} - x_2x_{2'}\hat{B} + x_{2'}\hat{D}) \mathcal{R}_{\ell_1,\ell_2}^{(\xi)}(u; x_1, x_2; x_1', x_2') = 0, \quad (6.16)$$

$$(x_1\hat{A} + \hat{C} - x_1x_{1'}\hat{B} + x_{1'}\hat{D}) \mathcal{R}_{\ell_1,\ell_2}^{(\xi)}(u; x_1, x_2; x_1', x_2') = 0.$$

The operator matrix in (6.15) is symmetric under the simultaneous exchange

$$1 \leftrightarrow 2', \quad 2 \leftrightarrow 1', \quad v \leftrightarrow -1 - u - v \quad (6.17)$$

and by this crossing symmetry the latter two conditions are transformed into each other.

We are going to solve the conditions (6.16) perturbatively in ξ up to the second order. There is no first order contribution,

$$\mathcal{R}^{(\xi)sym(\ell_1, \ell_2)} = \mathcal{R}^{[0]\ell_1, \ell_2} + \xi^2 \mathcal{R}^{[2]\ell_1, \ell_2} + \mathcal{O}(\xi^4), \quad (6.18)$$

since $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are symmetric in ξ . The kernel of the original non-symmetric operator is calculated from the symmetric one by

$$\mathcal{R}^{(\xi)(\ell_1, \ell_2)}(x_1, x_2; x_{1'}, x_{2'}) = e^{-u\xi(\partial_1 + \partial_{1'})} \mathcal{R}^{(\xi)sym(\ell_1, \ell_2)}(x_1, x_2; x_{1'}, x_{2'}). \quad (6.19)$$

The first condition (6.16) at order ξ^0 reads

$$\begin{aligned} & (v + \frac{1}{2} - \ell_2)[x_{2'1}x_{12}\partial_1 - (u + v + \frac{1}{2} + \ell_1)x_{12} - (u + v + \frac{1}{2} - \ell_1)x_{2'1}]\mathcal{R}^{[0]} \\ & + (v - \frac{1}{2} + \ell_2)[x_{1'2}x_{1'2'}\partial_{1'} - (u + v + \frac{3}{2} - \ell_1)x_{1'2'} - (u + v - \frac{1}{2} + \ell_1)x_{21'}]\mathcal{R}^{[0]} = 0. \end{aligned} \quad (6.20)$$

v appears here as a free parameter. The implication for $v = \ell_2 - \frac{1}{2}$ is

$$\partial_1 \left(\frac{x_{2'1}^{u+1+\ell_1-\ell_2}}{x_{12}^{u+1-\ell_1-\ell_2}} \right) \mathcal{R}^{[0]} = 0 \quad (6.21)$$

and for $v = \frac{1}{2} - \ell_2$

$$\partial_{1'} \left(\frac{x_{21'}^{u+1+\ell_2-\ell_1}}{x_{1'2'}^{u-1+\ell_1+\ell_2}} \right) \mathcal{R}^{[0]} = 0 \quad (6.22)$$

The two implications can be expressed together as

$$\partial_a \left(\frac{x_{2'1}^{u+1+\ell_1-\ell_2} x_{21'}^{u+1+\ell_2-\ell_1}}{x_{12}^{u+1-\ell_1-\ell_2} x_{1'2'}^{u-1+\ell_1+\ell_2}} \right) \mathcal{R}^{[0]} = 0 \quad (6.23)$$

for $a = 1, 1'$ The second condition in (6.16) implies in order ξ^0 the analogous condition for $a = 2, 2'$ which is clear from the mentioned crossing symmetry (6.17). In this way we recover the result for the kernel of the undeformed $sl(s)$ symmetric universal R operator.

$$\mathcal{R}^{[0]}(u) = c(u) \frac{x_{12}^{u-\ell_1-\ell_2+1} x_{1'2'}^{u+\ell_1+\ell_2-1}}{x_{2'1}^{u+1+\ell_1-\ell_2} x_{21'}^{u+1-\ell_1+\ell_2}}. \quad (6.24)$$

The first condition in (6.16) at order ξ^2 reads

$$\begin{aligned} & (v + \frac{1}{2} - \ell_2)[x_{2'1}x_{12}\partial_1 - (u + v + \frac{1}{2} + \ell_1)x_{12} - (u + v + \frac{1}{2} - \ell_1)x_{2'1}]\mathcal{R}^{[2]}(u) + \\ & + (v + \ell_2 - \frac{1}{2})[x_{1'2}x_{21'}\partial_{1'} - (u + v + \frac{3}{2} + \ell_1)x_{1'2'} - (u + v + \frac{1}{2} - \ell_1)x_{21'}]\mathcal{R}^{[2]}(u) + \\ & + (v + \frac{1}{2} - \ell_2)[\frac{1}{6}x_{12}\partial_1^3 x_{2'1} - \frac{1}{2}(u + v)\partial_1^2(x_{12} + x_{2'1}) + (u + v)(u + v + 1)\partial_1]\mathcal{R}^{[0]}(u) + \\ & + (v + \ell_2 - \frac{1}{2})[\frac{1}{6}x_{21'}\partial_{1'}^3 x_{1'2'} - \frac{1}{2}(u + v + 1)\partial_{1'}^2(x_{1'2'} + x_{21'}) + (u + v)(u + v + 1)\partial_{1'}]\mathcal{R}^{[0]}(u) + \\ & \frac{v}{2}\partial_2^2[x_{2'1}x_{12}\partial_1 - (u + v + \frac{1}{2} + \ell_1)x_{12} - (u + v + \frac{1}{2} - \ell_1)x_{2'1}]\mathcal{R}^{[0]}(u) + \end{aligned} \quad (6.25)$$

$$\begin{aligned}
& + \frac{v}{2} \partial_2^2 [x_{1'2'} x_{21'} \partial_{1'} - (u + v + \frac{3}{2} - \ell_1) x_{1'2'} - (u + v + \ell_1 - \frac{1}{2}) x_{21'}] \mathcal{R}^{[0]}(u) + \\
& + v^2 \partial_2 [u + v + \frac{1}{2} + \ell_1 - x_{2'1} \partial_1] R^{[0]}(u) + v^2 \partial_{2'} [u + v + \frac{3}{2} - \ell_1 - x_{21'} \partial_{1'}] \mathcal{R}^{[0]}(u) = 0.
\end{aligned}$$

The second condition in (6.16) leads to the analogous implication obtained from the latter by symmetry (6.17). By choosing $v = \pm(\frac{1}{2} - \ell_2)$ and using the result for $R^{[0]}(u)$ we obtain four differential equations for the function $R^{(2)}(u)$, which can be written together in the form

$$\begin{aligned}
& \partial_a [\mathcal{R}^{[2]}(u) / \mathcal{R}^{[0]}(u)] = \\
& = \partial_a \left[\frac{1}{12} (u+1+\ell_1-\ell_2)(u+2+\ell_1-\ell_2)(2u-\ell_1+\ell_2) \frac{1}{x_{2'1}^2} + \right. \\
& \quad \frac{2\ell_1-1}{4} (u+\ell_1+\ell_2-1)(u+1-\ell_1+\ell_2) \frac{1}{x_{21'} x_{1'2'}} \\
& \quad - \frac{1}{12} (u-\ell_1-\ell_2+1)(u-\ell_1-\ell_2)(2u+1+\ell_1+\ell_2) \frac{1}{x_{12}^2} - \frac{2\ell_1-1}{4} (u+1-\ell_1-\ell_2)(u+1+\ell_1-\ell_2) \frac{1}{x_{12} x_{2'1}} \\
& \quad - \frac{1}{12} (u+1-\ell_1-\ell_2)(u-\ell_1-\ell_2)(2u+1+\ell_1+\ell_2) \frac{1}{x_{12}^2} - \frac{2\ell_2-1}{4} (u-\ell_1-\ell_2+1)(u+1-\ell_1+\ell_2) \frac{1}{x_{12} x_{21'}} \\
& \quad - \frac{1}{12} (u+1-\ell_1+\ell_2)(u+2-\ell_1+\ell_2)(2u+\ell_1-\ell_2) \frac{1}{x_{21'}^2} + \frac{2\ell_2-1}{4} (u+\ell_1+\ell_2-1)(u+1+\ell_1-\ell_2) \frac{1}{x_{2'1} x_{1'2'}} \\
& \quad - \frac{1}{12} (u+\ell_1+\ell_2-1)(u-2+\ell_1+\ell_2)(2u+3-\ell_1-\ell_2) \frac{1}{x_{1'2'}^2} \\
& \quad \left. - \frac{2\ell_2-1}{4} (u-\ell_1-\ell_2+1)(u+1-\ell_1+\ell_2) \frac{1}{x_{12} x_{21'}} \right], \tag{6.26}
\end{aligned}$$

The index a labels the derivatives with respect to the points $x_1, x_2, x_{1'}, x_{2'}$. The ratio of the second order correction to the undeformed kernel is equal to the square bracket on the right hand side up to a constant. This constant can be absorbed by an unessential ξ^2 correction to $c(u)$ in (6.24). As the result up to the order ξ^2 the integral kernel of R^{sym} has form:

$$\begin{aligned}
& \mathcal{R}^{(\xi) sym(\ell_1, \ell_2)}(u) = e^{u\xi(\partial_1 + \partial_{1'})} \mathcal{R}^{(\xi)(\ell_1, \ell_2)}(u) = \\
& \mathcal{R}^{(0)(\ell_1, \ell_2)}(u) \left[1 - \frac{\xi^2}{12} [(2u+1+\ell_1+\ell_2)a_{11} - (2u+\ell_1-\ell_2)a_{22} \right. \\
& \quad \left. + (2u+3-\ell_1-\ell_2)a_{33} - (2u+\ell_2-\ell_1)a_{44}] \right. \\
& \quad \left. - \frac{\xi^2}{4} [(2\ell_1-1)(a_{14}-a_{23}) + (2\ell_2-1)(a_{12}-a_{34})] + \mathcal{O}(\xi^2) \right], \tag{6.27}
\end{aligned}$$

here we have introduced the notations

$$\begin{aligned}
a_{11} &= \frac{(u+1-\ell_1-\ell_2)(u-\ell_1-\ell_2)}{x_{12}^2}, & a_{22} &= \frac{(u+1+\ell_2-\ell_1)(u+2+\ell_2-\ell_1)}{x_{21'}^2}, \\
a_{33} &= \frac{(u+\ell_1+\ell_2-1)(u-2+\ell_1+\ell_2)}{x_{1'2'}^2}, & a_{44} &= \frac{(u+1+\ell_1-\ell_2)(u+2+\ell_1-\ell_2)}{x_{2'1}^2}, \\
a_{12} &= \frac{(u+1-\ell_1-\ell_2)(u+1+\ell_2-\ell_1)}{x_{12} x_{21'}}, & a_{13} &= \frac{(u+1-\ell_1-\ell_2)(u-1+\ell_1+\ell_2)}{x_{12} x_{1'2'}}, \\
a_{14} &= \frac{(u+1-\ell_1-\ell_2)(u+1+\ell_1-\ell_2)}{x_{12} x_{2'1}}, & a_{23} &= \frac{(u+1+\ell_2-\ell_1)(u+\ell_1+\ell_2-1)}{x_{21'} x_{1'2'}}, \\
a_{24} &= \frac{(u+1+\ell_2-\ell_1)(u+1+\ell_1-\ell_2)}{x_{21'} x_{2'1}}, & a_{34} &= \frac{(u-1+\ell_1+\ell_2)(u+1+\ell_1-\ell_2)}{x_{1'2'} x_{2'1}}.
\end{aligned}$$

7 Summary

Our study relies on the known algebraic structure of the ξ deformation of the $sl(2)$ Yangian as reviewed in Sect. 2. We have obtained the known fundamental representation R operator and the Lax operator both from the symmetry condition and from the Drinfeld twist transformation. We have written the Lax operator in two forms, one in terms of the deformed generators and one in terms of the undeformed generators. Therefrom we have established the relation of between the deformed and undeformed generators.

The defining condition of the universal R operator following from QYBE in the spirit of [13] has been formulated.

The structure of representations by polynomials in one variable and of the tensor product representations by polynomials in two variables has been investigated. We have written the deformed and undeformed generators in terms of several *Heisenberg* canonical pairs, the original one (y, ∂) and further ones expressed as non-trivial transformations of the former. The particular form of deformed generators, where S^- is the infinitesimal translation ∂ , is preferred in view of possible physical applications where momentum conservation is important. Then the twist relations between the deformed and undeformed generators induce a quite involved expression of the latter in terms of the original pair (x, ∂) .

The basis polynomials of the deformed representation and also the undeformed representation polynomials in the non-trivial induced form have been constructed. The resulting expressions involve products of the variable x displaced by subsequent units of the deformation parameter ξ . Also the polynomials of x_1, x_2 representing the lowest weight states of the irreducible representations in the tensor products have been calculated. They have been expressed in terms of products of the difference x_{12} with displacements depending on the representations ℓ_1, ℓ_2 and increasing in units of ξ and also summed up into a generating function.

With the explicit lowest weight polynomials we have studied the defining condition of the universal R operator writing the operators involved in terms of (x, ∂) . The resulting recurrence relation confirms the known fact that the eigenvalues of the R operator coincide with the ones of the undeformed R operator, as expected from the relation between them by the twist transformation. This transformation implies also a non-trivial relation between the series of polynomials in two variables representing the deformed tensor products and the induced form of the undeformed tensor products.

Finally the universal R operator in integral form has been considered. The defining conditions on the integral kernel have been solved perturbatively in ξ up to the first non-trivial order ξ^2 applying the projection method of [10].

We have treated the Jordanian deformation of the $sl(2)$ bi-algebra structure in view of physical applications where the undeformed $sl(2)$ represents infinitesimal conformal transformations with the dimensions of the representation being in general infinite. We have worked out these representations in terms of polynomial (wave) functions, generators and Lax operators in terms of differential operators and the universal R operator in spectral and integral forms in close analogy to the previously studied cases of supersymmetric extension ($sl(2|1)$) or standard q deformation of the $sl(2)$ bi-algebra structure. It will be interesting to study applications to integrable systems with conformal symmetry broken in such a way that translation symmetry is preserved.

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References.

- [1] L.N. Lipatov, *High-energy asymptotics of multicolor QCD and exactly solvable lattice models*, Padova preprint DFPD-93-TH-70B; and JETP Lett. B342 (1994)596.
- [2] L. D. Faddeev and G. P. Korchemsky, Phys. Lett. B **342** (1995) 311 [arXiv:hep-th/9404173].
- [3] H. J. De Vega and L. N. Lipatov, Phys. Rev. D**64** (2001) 114019, arXiv:hep-ph/0107225.
- [4] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, Nucl. Phys. B **617** (2001) 375, arXiv:hep-th/0107193.
- [5] V. M. Braun, S. E. Derkachov and A. N. Manashov, Phys. Rev. Lett. **81** (1998) 2020 [arXiv:hep-ph/9805225]; V. M. Braun, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, Nucl. Phys. B **553** (1999) 355 [arXiv:hep-ph/9902375].
- [6] A. V. Belitsky, Phys. Lett. B **453** (1999) 59 [arXiv:hep-ph/9902361]; Nucl. Phys. B **558** (1999) 259 [arXiv:hep-ph/9903512]; Nucl. Phys. B **574** (2000) 407 [arXiv:hep-ph/9907420].
- [7] N. Beisert and M. Staudacher, Nucl. Phys. B **670** (2003) 439 [arXiv:hep-th/0307042].
- [8] L. Dolan, C. R. Nappi and E. Witten, arXiv:hep-th/0308089.
- [9] S. Derkachov, D. Karakhanian and R. Kirschner, Nucl. Phys. B **583** (2000) 691 [arXiv:nlin.si/0003029].
- [10] S. E. Derkachov, D. Karakhanian and R. Kirschner, Nucl. Phys. B **618** (2001) 589, arXiv:nlin.si/0102024.
- [11] D. Karakhanian, R. Kirschner and M. Mirumyan, Nucl. Phys. B **636** (2002) 529 [arXiv:nlin.si/0111032].
- [12] P.P. Kulish and E.K. Sklyanin, Zap. Nauchn. Semin. LOMI 95 (1980) 129.
- [13] V.O. Tarasov, L.A. Takhtadjan and L.D. Faddeev, Theor. Math. Phys. 57 (1983) 163-181.
- [14] E.K. Sklyanin, "Quantum Inverse Scattering Method", in *Quantum Groups and Quantum Integrable Systems*, (Nankai lectures), ed. Mo-Lin Ge, pp. 63-97, World Scientific Publ., Singapore 1992, [hep-th/9211111]
- [15] L.D. Faddeev, Les Houches lectures 1995, hep-th/9605187.

- [16] E.E. Demidov, Yu.I. Manin, E.E. Mukhin and D.V. Zhdanovich, *Progr. Theor. Phys. Suppl.* **102** (1990) 203.
- [17] S. Zakrewski, *Lett. Math. Phys.* **22** (1991) 287.
- [18] A. Stolin, *Math. Scand.* **69** (1991) 56.
- [19] H. Ewen, O.V. Ogievetsky and J. Wess, *Lett. Math. Phys.* **22** (1991) 297; H. Ewen and O. Ogievetsky, *Jordanian solutions of simplex equations*, hep-th/9211026
- [20] B.A. Kupershmidt, *J Phys A Math Gen* **25** (1992) L1239-L1244.
- [21] Ch. Ohn, *Lett. Math. Phys.* **25** (1992) 85.
- [22] M. Gerstenhaber, A. Giaquinto, S.D. Schack, in *Quantum Groups, Proceedings In EIMI*, P. Kulish (ed.), *Lecture Notes in Math.*, No.1510, p. 9, Springer 1992.
- [23] O.V. Ogievetsky, *Suppl. Rendiconti Cir. Math. Palermo, Serie II* **37** (1993) 185.
- [24] V.G. Drinfeld, *Soviet Math. Dokl.* **27** (1983) 68.
- [25] V.G. Drinfeld, *Soviet Math. Dokl.* **32** (1985) 254 and *ibid.* **36** (1985) 212
- [26] P.P. Kulish and A. A Stolin, *Czech. J. Phys.* **47** (1997) 123: and 1207. q-alg/9608011; and q-alg/9708024.
- [27] S. Khoroshkin, A. Stolin, V. Tolstoy, in: *From Field Theory to Quantum Groups*, eds. B. Jancewicz and J. Sobczyk, World Scientific (1996), pp. 53-77. q-alg/9511005.
- [28] B. Abdesselam, A. Chakrabarti, R. Chakrabarti, *Mod. Phys. Lett. A* **11** (1996) 2883.
- [29] J. Van der Jeugt, *J. Phys. A: Math. Gen.* **31** (1998) 1495-1508, q-alg/9703011; *Czech. J. Phys.* **47** (1997) 1283-1289, q-alg/9709005.
- [30] A.A. Vladimirov, *Mod. Phys. Lett. A* **8** (1993) 2573.
- [31] A. Ballesteros, F.J. Herranz, M.A. del Olmo, M. Santander, *Journ. Math. Phys.* **35** (1994) 4928; and *Phys. Lett B* **351** (1995) 137.
- [32] J. Lukierski, P. Minnaret and M. Morzrzymas, *Phys. Lett. B* **371** (1996) 215, q-alg/9507005;
J. Lukierski, 'From noncommutative space-time to quantum relativistic symmetries with fundamental mass parameter,' arXiv:hep-th/0112252;
J. Lukierski, V. Lyakhovsky and M. Mozrzymas, *Phys. Lett. B* **538** (2002) 375 [arXiv:hep-th/0203182].
- [33] A.A. Belavin and V.G. Drinfeld, *Funct. Anal. Appl.* **16** (1982) No.3, 1-29.

8 Appendices

8.1 Appendix A

We show that the deformed commutation relations follow from the relations (3.19) of the deformed and undeformed generators. The following commutators with functions of S_0^- are useful,

$$[S_0^0, f(S_0^-)] = -\alpha \partial_\alpha f(\alpha)|_{\alpha=S_0^-}, \quad (8.1)$$

$$[S_0^+, f(S_0^-)] = 2\partial_\alpha f(\alpha) S_0^0 - \alpha \partial_\alpha^2 f(\alpha)|_{\alpha=S_0^-}, \quad (8.2)$$

$$[S_0^0(S_0^0 + \frac{1}{2}), f(S_0^-)] = -2\alpha \partial_\alpha f(\alpha) S_0^0 + (\alpha^2 \partial_\alpha^2 + \frac{1}{2}\alpha \partial_\alpha) f(\alpha)|_{\alpha=S_0^-}, \quad (8.3)$$

They imply in particular,

$$[S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})] \sqrt{1 - 2\xi S_0^-} = \sqrt{1 - 2\xi S_0^-} [S_0^+ + 2\xi S_0^0(S_0^0 - \frac{1}{2})] \quad (8.4)$$

$$\left[S_0^0, \sqrt{1 - 2\xi S_0^-} \right] = \frac{\xi S_0^-}{\sqrt{1 - 2\xi S_0^-}} \quad (8.5)$$

Let us check first the relation

$$[S_\xi^0, S_\xi^-] = -\frac{1}{\xi} \text{sh}(\xi S_\xi^-) \quad (8.6)$$

We substitute the expressions in terms of the undeformed generators (3.19). The right-hand side is equal to $-S_0^-(1 - 2\xi S_0^-)^{-\frac{1}{2}}$. The commutator on the left-hand side is calculated by using (8.1) and the result is easily seen to coincide with the one for the right-hand side.

Let us check now the relation

$$[S_\xi^+, S_\xi^-] = \frac{1}{2} \{ \text{ch}(\xi S_\xi^-), S_\xi^+ \} \quad (8.7)$$

We write the right-hand side in terms of the undeformed generators and apply (8.4),

$$\begin{aligned} & \frac{1}{4} \left(\frac{1}{\sqrt{1 - 2\xi S_0^-}} + \sqrt{1 - 2\xi S_0^-} \right) [S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})] \sqrt{1 - 2\xi S_0^-} + \\ & \frac{1}{4} [S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})] \sqrt{1 - 2\xi S_0^-} \left(\frac{1}{\sqrt{1 - 2\xi S_0^-}} + \sqrt{1 - 2\xi S_0^-} \right) = \\ & \frac{1 - \xi S_0^-}{2} [S_0^+ + 2\xi S_0^0(S_0^0 - \frac{1}{2})] + [S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})] \frac{1 - \xi S_0^-}{2} = \\ & S_0^+ + \xi \left(2(S_0^0)^2 - \frac{1}{2}(S_0^+ S_0^- + S_0^- S_0^+) \right) - \xi^2 \left(S_0^- (S_0^0)^2 + (S_0^0)^2 S_0^- - \frac{1}{2} S_0^- \right). \end{aligned}$$

We write also the left-hand side in terms of the undeformed generators and apply (8.4, 8.5),

$$-\frac{1}{2} \xi S_0^- [S_0^+ + 2\xi S_0^0(S_0^0 - \frac{1}{2})] + S_0^0(1 - 2\xi S_0^-) [S_0^+ + 2\xi S_0^0(S_0^0 - \frac{1}{2})] +$$

$$\begin{aligned}
& [S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})]\frac{1}{2}\xi S_0^- - [S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2})](S_0^0 + \xi S_0^-(1 - 2S_0^0)) = \\
& S_0^0 S_0^+ - S_0^+ S_0^0 + \xi[-\frac{1}{2}S_0^- S_0^+ + 2(S_0^0)^2(S_0^0 - \frac{1}{2}) - 2S_0^0 S_0^- S_0^+ + \frac{1}{2}S_0^+ S_0^- - 2(S_0^0)^2(S_0^0 + \frac{1}{2})] - S_0^+ S_0^-(1 - 2S_0^0) + \\
& \xi^2[-S_0^- S_0^0(S_0^0 - \frac{1}{2}) - 4S_0^0 S_0^- S_0^0(S_0^0 - \frac{1}{2}) + S_0^0(S_0^0 + \frac{1}{2})S_0^- - S_0^0(1 + 2S_0^0)S_0^-(1 - 2S_0^0)]
\end{aligned}$$

Now it is easy to check that the coefficients of ξ^2, ξ^1, ξ^0 of both sides are equal. The right hand side of the third commutator is equal to

$$-\frac{\xi S_0^-}{\sqrt{1 - 2\xi S_0^-}} + 2S_0^0 \sqrt{1 - 2\xi S_0^-}$$

The commutator on the left-hand side is

$$-\frac{1}{2\xi}[S_0^+ + 2\xi S_0^0(S_0^0 + \frac{1}{2}), \ln(1 - 2\xi S_0^-)]\sqrt{1 - 2\xi S_0^-}$$

We apply (8.2, 8.3). Then it is easy to see that the result is equal to the right hand side.

8.2 Appendix B

We list the operator valued matrix elements of $K_L(u)$ and $K_R(u)$ introduced in (3.23).

$$\begin{aligned}
\xi K_L^{11} &= (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- + S_{\xi,1}^0)(\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0) + \frac{1}{\xi}\text{sh}\xi S_{\xi,1}^- S_{\xi,2}^+ + \\
&+ \frac{u}{2} \left(e^{-\xi S_{\xi,1}^-} (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0) - (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- + S_{\xi,1}^0) e^{-\xi S_{\xi,2}^-} + \text{sh}\xi S_{\xi,1}^- (2S_{\xi,2}^0 + \text{sh}\xi S_{\xi,2}^-) \right), \\
\xi K_L^{12} &= \frac{u}{2} \left(e^{-\xi S_{\xi,1}^-} \text{sh}\xi S_{\xi,2}^- - \text{sh}\xi S_{\xi,1}^- e^{\xi S_{\xi,2}^-} \right) + (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- + S_{\xi,1}^0) \text{sh}\xi S_{\xi,2}^- + \text{sh}\xi S_{\xi,1}^- (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- - S_{\xi,2}^0), \\
\xi K_L^{22} &= (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- - S_{\xi,1}^0)(\frac{1}{2}\text{ch}\xi S_{\xi,2}^- - S_{\xi,2}^0) + \frac{1}{\xi}\text{sh}\xi S_{\xi,2}^- S_{\xi,1}^+ + \\
&+ \frac{u}{2} \left(e^{\xi S_{\xi,1}^-} (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- - S_{\xi,2}^0) - (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- + S_{\xi,1}^0) e^{\xi S_{\xi,2}^-} + \text{sh}\xi S_{\xi,2}^- (2S_{\xi,1}^0 + \text{sh}\xi S_{\xi,1}^-) \right), \\
\xi K_L^{21} &= S_{\xi,1}^+ (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0) + (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- - S_{\xi,1}^0) S_{\xi,2}^+ + \\
&+ \frac{u}{2} \left(\xi (2S_{\xi,1}^0 + \text{sh}\xi S_{\xi,1}^-) (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0) + e^{\xi S_{\xi,1}^-} S_{\xi,2}^+ - e^{-\xi S_{\xi,2}^-} S_{\xi,1}^+ - \xi (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- - S_{\xi,1}^0) (2S_{\xi,2}^0 + \text{sh}\xi S_{\xi,2}^-) \right)
\end{aligned}$$

The expressions for the elements of right matrix are similar

$$\begin{aligned}
\xi K_R^{11} &= (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- + S_{\xi,1}^0)(\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0) + \frac{1}{\xi}\text{sh}\xi S_{\xi,2}^- S_{\xi,1}^+ + \\
&+ \frac{u}{2} \left(e^{-\xi S_{\xi,1}^-} (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0) - (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- + S_{\xi,1}^0) e^{-\xi S_{\xi,2}^-} + \text{sh}\xi S_{\xi,2}^- (2S_{\xi,1}^0 + \text{sh}\xi S_{\xi,1}^-) \right), \\
\xi K_R^{12} &= \frac{u}{2} \left(e^{\xi S_{\xi,1}^-} \text{sh}\xi S_{\xi,2}^- - \text{sh}\xi S_{\xi,1}^- e^{-\xi S_{\xi,2}^-} \right) + (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- - S_{\xi,1}^0) \text{sh}\xi S_{\xi,2}^- + \text{sh}\xi S_{\xi,1}^- (\frac{1}{2}\text{ch}\xi S_{\xi,2}^- + S_{\xi,2}^0), \\
\xi K_R^{22} &= (\frac{1}{2}\text{ch}\xi S_{\xi,1}^- - S_{\xi,1}^0)(\frac{1}{2}\text{ch}\xi S_{\xi,2}^- - S_{\xi,2}^0) + \frac{1}{\xi}\text{sh}\xi S_{\xi,1}^- S_{\xi,2}^+ +
\end{aligned}$$

$$\begin{aligned}
& + \frac{u}{2} \left(e^{\xi S_{\xi,1}^-} \left(\frac{1}{2} \text{ch} \xi S_{\xi,2}^- - S_{\xi,2}^0 \right) - \left(\frac{1}{2} \text{ch} \xi S_{\xi,1}^- + S_{\xi,1}^0 \right) e^{\xi S_{\xi,2}^-} + \text{sh} \xi S_{\xi,1}^- (2S_{\xi,2}^0 + \text{sh} \xi S_{\xi,2}^-) \right), \\
& \xi K_R^{21} = S_{\xi,1}^+ \left(\frac{1}{2} \text{ch} \xi S_{\xi,2}^- - S_{\xi,2}^0 \right) + \left(\frac{1}{2} \text{ch} \xi S_{\xi,1}^- + S_{\xi,1}^0 \right) S_{\xi,2}^+ + \\
& + \frac{u}{2} \left(\xi (2S_{\xi,1}^0 + \text{sh} \xi S_{\xi,1}^-) \left(\frac{1}{2} \text{ch} \xi S_{\xi,2}^- - S_{\xi,2}^0 \right) + e^{-\xi S_{\xi,1}^-} S_{\xi,2}^+ - e^{\xi S_{\xi,2}^-} S_{\xi,1}^+ - \xi \left(\frac{1}{2} \text{ch} \xi S_{\xi,1}^- + S_{\xi,1}^0 \right) (2S_{\xi,2}^0 + \text{sh} \xi S_{\xi,2}^-) \right)
\end{aligned}$$

8.3 Appendix C

The eigenvalue condition (4.8) can be written as

$$\frac{1}{2} (x \partial_x^\xi + x \partial_x^\xi) \varphi^{(m)}(x) = (m + \frac{1}{2}) \varphi^{(m)}(x) \quad (8.8)$$

It is actually independent of ℓ . We calculate the action of the operator on l.h.s. on products,

$$\begin{aligned}
\frac{1}{2} (x \partial_x^\xi + x \partial_x^\xi) \prod_{k=-m_1}^{m_1} (x + k\xi) &= (2m_1 + \frac{3}{2}) \prod_{k=-m_1}^{m_1} (x + k\xi) \\
&+ \xi^2 2m_1 (m_1 + \frac{1}{2})^2 \prod_{k=-m_1+1}^{m_1-1} (x + k\xi), \\
\frac{1}{2} (x \partial_x^\xi + x \partial_x^\xi) x \prod_{k=-m_1}^{m_1} (x + k\xi) &= (2m_1 + \frac{5}{2}) x \prod_{k=-m_1}^{m_1} (x + k\xi) \\
&+ \xi^2 2(m_1 + 1) (m_1 + \frac{1}{2})^2 x \prod_{k=-m_1+1}^{m_1-1} (x + k\xi). \quad (8.9)
\end{aligned}$$

From these relation we understand that the polynomial eigenfunctions obeying (8.8) are finite sums of such products,

$$\begin{aligned}
\varphi^{(2m_1+1)}(x) &= \sum_{k=0}^{m_1} a_k^{(m_1)} \xi^{2k} \prod_{k_1=-m_1+k}^{m_1-k} (x + k_1 \xi), \\
\varphi^{(2m_1+2)}(x) &= \sum_{k=0}^{m_1+1} b_k^{(m_1)} \xi^{2k} x \prod_{k_1=-m_1+k}^{m_1-k} (x + k_1 \xi). \quad (8.10)
\end{aligned}$$

Substituting this into (8.8) leads to simple iterative relations for the coefficients with the solution

$$\begin{aligned}
a_k^{(m_1)} &= \binom{m_1}{k} \left(\frac{(2m_1+1)!!}{(2(m_1-k)+1)!! 2^k} \right)^2, \\
b_k^{(m_1)} &= \binom{m_1+1}{k} \left(\frac{(2m_1+1)!!}{(2(m_1-k)+1)!! 2^k} \right)^2. \quad (8.11)
\end{aligned}$$

To derive the generating function $G_1(x, t)$ (4.12) we write $\varphi_\ell^{(m)}$ as a Fourier integral and $\varphi_\ell^{(m)} = (X_J)^m \varphi_\ell^{(0)}$ as

$$\varphi^{(m)}(x) = \frac{1}{2^m} \left(x(1 + \text{ch} \xi \partial) + \frac{\xi}{2} \text{sh} \xi \partial \right)^m \int dk a_0(k) e^{ikx}. \quad (8.12)$$

The Fourier image $a_0(k)$ of $\varphi_0(x) = 1$ is given by delta-function $\delta(k)$. Now it is convenient to do the change of variables $\alpha = \tan(k\xi/2)$,

$$\left(x(1 + \cos k\xi)\frac{d}{dk} - \frac{1}{2}\sin k\xi\right)^n = (\cos(k\xi/2))^{-1}(\partial/\partial\alpha)^n \cos(k\xi/2).$$

Then the series of the generating function can be summed up using *Taylor's* formula

$$\begin{aligned} G_1(x, t) &= \int dk \frac{a_0(k)}{\cos k\xi/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t\xi}{2i} \frac{\partial}{\partial\alpha}\right)^n e^{ikx} \cos k\xi/2 = \\ &= \int dk a_0(k) \sqrt{1 + \alpha^2(k)} \left(\frac{1 + i(\alpha + \frac{t\xi}{2i})}{1 - i(\alpha + \frac{t\xi}{2i})}\right)^{\frac{x}{\xi}} \left(1 + i(\alpha + \frac{t\xi}{2i})\right)^{-\frac{1}{2}} = \\ &\quad \left(\frac{1 + \frac{t\xi}{2}}{1 - \frac{t\xi}{2}}\right)^{\frac{x}{\xi}} \left(1 + \frac{t\xi}{2}\right)^{-\frac{1}{2}} \left(1 - \frac{t\xi}{2}\right)^{-\frac{1}{2}} = \frac{\left(1 + \frac{t\xi}{2}\right)^{\frac{x}{\xi} - \frac{1}{2}}}{\left(1 - \frac{t\xi}{2}\right)^{\frac{x}{\xi} + \frac{1}{2}}}. \end{aligned} \tag{8.13}$$